A characteristic map for the symmetric space of symplectic forms over a finite field

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Introduction

There is an isomorphism between the class functions on the symmetric groups and symmetric functions. This has been extended to $\mathrm{GL}_n(\mathbf{F}_q)$ and the Gelfand pair S_{2n}/B_n. We develop an analogous theory for $\mathrm{GL}_{2n}(\mathbf{F}_q)/\mathrm{Sp}_{2n}(\mathbf{F}_q)$ and use it to study the Schur expansion of Macdonald polynomials.

Gelfand pairs

A **Gelfand pair** is a pair $H \subseteq G$ of groups such that convolution on bi-invariant functions $\mathbf{C}[H\backslash G/H]$ is commutative. The theory of Gelfand pairs parallels that of groups:

- H-**bi-invariant functions** correspond to class functions.
- **Double cosets** correspond to conjugacy classes.
- **Spherical functions** correspond to irreducible characters.

The symmetric space $\mathrm{GL}_{2n}(\mathbf{F}_q)/\mathrm{Sp}_{2n}(\mathbf{F}_q)$

The symplectic group $H = \mathrm{Sp}_{2n}(\mathbf{F}_q)$ sitting inside $G = \mathrm{GL}_{2n}(\mathbf{F}_q)$ is the set of fixed points of an automorphism ι, and form a Gelfand pair. Its theory parallels that of $\mathrm{GL}_n(\mathbf{F}_q)$:

- Double cosets $Hg_\mu H$ are indexed by partition-valued functions μ : {irreducible polynomials/$\mathbf{F}_q$} $\to \mathcal{P}$.

Example: Let $n = 2$, $q = 2$. The conjugacy classes of $\mathrm{GL}_2(\mathbf{F}_q)$ are labeled by the functions $\mu_1(x-1) = (11)$, $\mu_2(x-1) = (2)$ and $\mu_3(x^2+x+1) = (1)$ (all other polynomials map to $\emptyset$). These correspond to rational canonical forms

$$J_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad J_3 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

The corresponding double cosets have representatives

$$g_{\mu_i} = \begin{pmatrix} J_i & 0 \\ 0 & I \end{pmatrix} \qquad i = 1, 2, 3.$$

- Spherical functions ϕ_λ are indexed by partition-valued functions λ : {cuspidal representations} $\to \mathcal{P}$.

Induction and restriction

Let P be a rational ι-stable parabolic subgroup with rational ι-stable Levi factor L and unipotent radical U. Define the following two operations on bi-invariant functions:

- **Bi-invariant parabolic induction:**
$$\mathrm{Ind}_{L\subseteq P}^{G/H}(f)(x) = |H \cap P|^{-2} \sum f(\overline{hxh'}).$$
- **Bi-invariant parabolic restriction:**
$$\mathrm{Res}_{L\subseteq P}^{G/H}(f)(x) := \sum_{p \in P, \overline{p}=x} f(p).$$

They satisfy the following properties:

- **Conjugation-invariant:** Invariant under conjugation of $L \subseteq P$ by H.
- **Transitive:** $\mathrm{Ind}_{M\subseteq Q}^{G/H} = \mathrm{Ind}_{L\subseteq P}^{G/H} \circ \mathrm{Ind}_{M\subseteq Q\cap L}^{L/H\cap L}$ and similarly for Res.
- **Adjoint:** $\langle f, \mathrm{Ind}_{L\subseteq P}^{G/H} g\rangle = |H|^2 |H \cap P|^{-2} \langle \mathrm{Res}_{L\subseteq P}^{G/H} f, g\rangle$.

Induction gives a way to multiply a class function on $\mathrm{GL}_m(\mathbf{F}_q)$ and a bi-invariant function on $\mathrm{GL}_{2n}(\mathbf{F}_q)$, using Levi and parabolic subgroups $L \subseteq P \subseteq \mathrm{GL}_{2n+2m}(\mathbf{F}_q)$

$$L = \left\{ \begin{pmatrix} * & & & \\ & * & * & \\ & * & * & \\ & & & * \end{pmatrix} \right\}, \quad P = \left\{ \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & * & * & * \\ & & & * \end{pmatrix} \right\}.$$

The characteristic map

The **characteristic map** is a function

$$\mathrm{ch} : \bigoplus_n \mathbf{C}[\mathrm{Sp}_{2n}(\mathbf{F}_q)\backslash \mathrm{GL}_{2n}(\mathbf{F}_q)/\mathrm{Sp}_{2n}(\mathbf{F}_q)] \to \bigotimes_f \Lambda$$

where Λ is the ring of symmetric functions, defined by

$$\mathrm{ch}(I_{Hg_\mu H}) = \prod_f q_f^{-2n(\mu(f))} P_{\mu(f)}(f; q_f^{-2})$$

and extending linearly. Here, the $P_\lambda(t)$ denote the **Hall-Littlewood polynomials**.

Theorem: The collection of all $\mathrm{Sp}_{2n}(\mathbf{F}_q)$-bi-invariant functions on $\mathrm{GL}_{2n}(\mathbf{F}_q)$ for $n \in \mathbf{N}$ is isomorphic to a tensor power of the ring of symmetric functions under the characteristic map ch. The map ch has the following properties:

- **Isometry:** Up to a scalar, ch takes the inner product of bi-invariant functions to the (q, q^2) Macdonald inner product.
- **Product:** The bi-invariant functions form a module over the class functions for $\mathrm{GL}_n(\mathbf{F}_q)$ under a bi-invariant version of parabolic induction, and ch preserves this structure up to a twist.
- **Spherical functions:** Up to a scalar, ch takes the spherical functions to Macdonald polynomials of parameter (q, q^2).

Schur expansion of Macdonald Polynomials

The skew Macdonald polynomial $P_{\lambda/\mu}(x; q, t)$ is defined by requiring $\langle P_{\lambda/\mu}(q,t), f\rangle = \langle P_\lambda(q,t), Q_\mu(q,t) f\rangle$ for all symmetric functions f.

Theorem: If q is an odd prime power, then the skew-Macdonald polynomials $P_{\lambda/\mu}(q, q^2)$ expand positively into Schur functions.

Our application is related to a conjecture of Haglund which states that

$$(1-q)^{-|\lambda|} \langle J_\lambda(x; q, q^k), s_\mu\rangle \in \mathbf{N}[q].$$

In fact, some computations done in Sage suggest the following extension. Let $J_\mu^\perp(q, q^2)$ denote the adjoint to multiplication by $J_\mu(q, q^2)$ under the (q, q^2)-inner product. Then it seems that

$$(1-q)^{-|\lambda|-|\mu|} \langle J_\mu^\perp(q, q^2) J_\lambda(q, q^2), s_\nu\rangle \in \mathbf{N}[q].$$

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References

[1] J. He. A characteristic map for the symmetric space of symplectic forms over a finite field, 2019. arXiv:1906.05966.